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ANALYTICAL ANALYSIS OF MOVING FRONTS IN TWO-AND THREE-DIMENSIONAL GROUNDWATER FLOW IN ANISOTROPIC AQUIFERS

1.0. Introduction.

An important object of study at many institutes in the world, dealing with groundwater problems, nowadays is the analysis of the movement of fronts between groundwater bodies of different quality by means of analytical or numerical calculation methods. This is especially understandable for the Netherlands, where groundwater is mainly exploited from thick pleistocene aquifers in the eastern and southern part of the country where brackish or salt water is practically always found in the deeper regions.
Extraction of groundwater from such an aquifer by means of pumping wells is always accompanied by the risk of attracting the brackish water with the possibility of salination of the pumped fresh water.
Of great importance in this case is (therefore) an answer to the question how long it will last, if pumping goes on with a certain capacity, until the brackish water from the deeper layers will reach the well (if it does), in such a way that the pumped mixture of fresh and brackish water has attained too high a chlorine content for drinking water purposes.
In its most simple form the problem is reduced to a single well of a certain length in an infinite aquifer with a brackish/fresh water interface at a fixed distance (fig. 1).

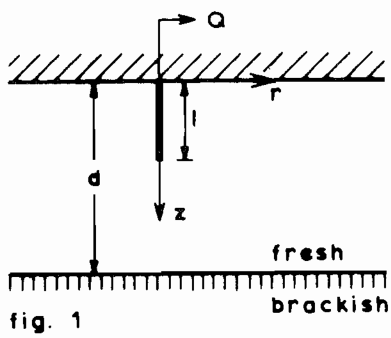

fig. 1

It is assumed that the interface is horizontal at the beginning of the pumping, the discharge of the well being Q m^3 per hour.
The main problem is how to predict the rising of the interface, caused by the groundwater withdrawal. Even if the dispersion and the difference in density between fresh and salt or brackish water is neglected, this problem cannot easily be solved. The problem is a so called axial sym-

metric three-dimensional problem and both a horizontal radial component v_r and a vertical component v_z of the velocity must be considered.
The first step in solving this problem is to determine the stream- and potential-lines governing the flow problem and most probably a difference between the horizontal and vertical permeabilities of the soil (anisotropy) has to be taken into account.
The next step is to determine the change in form of the initial horizontal interface with time as a consequence of the flow pattern caused by the groundwater extraction. Here the greatest difficulties are met if one wants the solution of the problem in an analytical form.
In most cases numerical or graphical solutions can be found or solutions be obtained by means of analogons. In fact, for complicated problems only these methods may lead to an acceptable result, but analytical solutions, if they can be obtained in such a way that they are fit for practical use, are preferable.
The aim of this paper is to show how, in certain cases, analytical solutions for moving fronts in two- and three-dimensional groundwater flow can be obtained which can be used in practice to get a general insight in the problem of attracting groundwater with a higher Cl^- content or an otherwise different quality.

2.0. Partially penetrating well in a deep aquifer.

2.1. Analysis of the flow problem.

If the total discharge of the well is assumed to be uniformly distributed along the well screen, one can find the drawdown distribution by integrating the drawdown caused by a point-source along the length of the well.
The drawdown of a point-source is equal to

$$\varphi = \frac{Q}{4\pi K\rho} = \frac{Q}{4\pi K\sqrt{r^2+z^2}}$$

and for a source in the point P (o,z_o) with strength $\frac{2Q}{2l}$:

$$d\varphi = \frac{Q dz_o}{4\pi K l\sqrt{r^2+(z-z_o)^2}}$$

Integration from - l till + l yields for the total drawdown:

$$Q = \frac{Q}{4\pi Kl} \int_{-l}^{+l} \frac{dz_o}{\sqrt{r^2+(z-z_o)^2}} \qquad \text{and so}$$

(1) $$\varphi(r,z) = \frac{Q}{4\pi Kl} \left\{ \text{arcsinh}\,(\frac{l+z}{r}) + \text{arcsinh}\,(\frac{l-z}{r}) \right\}$$

which is called the potential function of the problem.
To find the stream function Ψ we make use of the relations

$$v_r = K \frac{d\varphi}{dr} = \frac{1}{2\pi r} \frac{d\Psi}{dz} \quad \text{and}$$

$$v_z = K \frac{d\varphi}{dz} = - \frac{1}{2\pi r} \frac{d\Psi}{dr}$$

which leads to:

(2) $$\Psi(r,z) = - \frac{Q}{2l}\left\{\sqrt{r^2+(l+z)^2} - \sqrt{r^2+(l-z)^2}\right\}$$

Ψ is defined so that it varies from o along the r-axis (z=o) to Ψ = - Q along the z-axis (r=o) for z > l
(for z > l and r=o one has to replace l - z by z - l in formula 2)
From formula 2 follows that

$$4l^2 \frac{\Psi^2}{Q^2} = r^2+(l+z)^2+r^2+(l-z)^2-2 \sqrt{(r^2+z^2+l^2+2lz)(r^2+z^2+l^2-2lz)}$$

$$= 2(r^2+z^2+l^2-2\sqrt{(r^2+z^2+l^2)^2 - 4l^2z^2}$$

$$(r^2+z^2+l^2)^2-4l^2z^2 = (r^2+z^2+l^2)^2-4l^2(r^2+z^2+l^2) \frac{\Psi^2}{Q^2} + 4l^4 \frac{\Psi^4}{Q^4}$$

$$(1- \frac{\Psi^2}{Q^2}) z^2- \frac{\Psi^2}{Q^2} r^2 = \frac{l^2\Psi^2}{Q^2} (1- \frac{\Psi^2}{Q^2})$$

Dividing by $\frac{l^2\Psi^2}{Q^2} (1 - \frac{\Psi^2}{Q^2})$ one obtains:

(3) $$\frac{z^2}{\frac{l^2\Psi^2}{Q^2}} - \frac{r^2}{l^2(1 - \frac{\Psi^2}{Q^2})} = 1,$$

the equation for the stream surfaces in the form of a family of confocal hyperboloids of revolution with common focus F (o,l) and the coordinate axis as main axis.

The equipotential surfaces form a family of surfaces of revolution, orthogonal to the stream surfaces and consist of confocal ellipsoids of revolution with the same common focus F (o,l).

Its formula is:

$$\frac{r^2}{a^2} + \frac{z^2}{b^2} = 1 \text{ with } b^2 - a^2 = l^2$$

Now a and b are defined by the points of intersection of the ellipsoids with the coordinate axis.

Therefore one may derive from formula 1:

$$\varphi(a,o) = \frac{Q}{2\pi Kl} \operatorname{arcsinh} \left(\frac{l}{a}\right) \text{ from which}$$

$$a = \frac{l}{\sinh\left(\frac{2\pi Kl\varphi}{Q}\right)} = l \operatorname{cosech}\left(\frac{2\pi lK\varphi}{Q}\right)$$

$$b^2 = l^2 + a^2 = l^2 (1 + \operatorname{cosech}^2) = l^2 \operatorname{cotgh}^2$$

$$b = l \operatorname{cotgh}\left(\frac{2\pi lK\varphi}{Q}\right)$$

This result also follows from

$$\varphi(o,b) = \frac{Q}{4\pi Kl} \ln\left(\frac{b+l}{b-l}\right) = \frac{Q}{2\pi Kl} \operatorname{arctgh}\left(\frac{l}{b}\right)$$

bearing in mind that $\operatorname{arcsinh} \alpha = \ln(\alpha + \sqrt{\alpha^2+1})$

One thus obtains for the formula of the equipotential surfaces:

$$(4) \quad \frac{r^2}{l^2 \operatorname{cosech}^2\left(\frac{2\pi lK\varphi}{Q}\right)} + \frac{z^2}{l^2 \operatorname{cotgh}^2\left(\frac{2\pi Kl\varphi}{Q}\right)} = 1$$

See fig. 2 and also appendix 1.

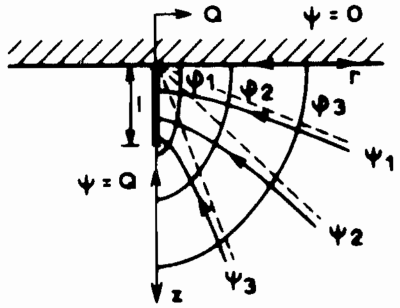

fig. 2

2.2. General equation of motion.

Along a fixed stream-surface Ψ_o the equation of motion can be written as

$$v = \varepsilon \frac{ds}{dt} \text{ or } t = \varepsilon \int^{\Psi_o} \frac{ds}{v} \text{ in which}$$

ε is the so-called effective porosity of the soil.

For practical use v and ds are expressed in the curvilinear coordinates φ and Ψ :

$$v = \sqrt{v_r^2 + v_z^2} = K \sqrt{\left(\frac{d\varphi}{dr}\right)^2 + \left(\frac{d\varphi}{dz}\right)^2}$$

$$ds = \sqrt{(dr)^2 + (dz)^2} = \sqrt{\left(\frac{dr}{d\varphi}\right)^2 + \left(\frac{dz}{d\varphi}\right)^2} \, . \, d\varphi \quad \text{for } \Psi = \text{constant.}$$

One can express $\frac{d\varphi}{dr}$ and $\frac{d\varphi}{dz}$ in

$\frac{dr}{d\varphi}$ and $\frac{dz}{d\varphi}$ as follows:

φ and Ψ are functions of r and z :

$\varphi = \varphi\,(r,z)$ and $\Psi = \Psi\,(r,z)$

Along a stream surface, Ψ is a constant, whence:

$$d\varphi = \frac{d\varphi}{dr}\,dr + \frac{d\varphi}{dz}\,dz \text{ or}$$

$$1 = \frac{d\varphi}{dr} \cdot \frac{dr}{d\varphi} + \frac{d\varphi}{dz} \cdot \frac{dz}{d\varphi} \tag{a}$$

$$d\Psi = o = \frac{d\Psi}{dr}\,dr + \frac{d\Psi}{dz}\,dz = -\,2\pi rK \frac{d\varphi}{dz}\,dr + 2\pi rK \frac{d\varphi}{dr}\,dz \quad \text{or}$$

$$o = \frac{d\varphi}{dr}\,\frac{dz}{d\varphi} - \frac{d\varphi}{dz}\,\frac{dr}{d\varphi} \tag{b}$$

$\frac{d\varphi}{dr}$ and $\frac{d\varphi}{dz}$ solved from (a) and (b) gives:

$$\frac{d\varphi}{dr} = \frac{\frac{dr}{d\varphi}}{(\frac{dr}{d\varphi})^2 + (\frac{dz}{d\varphi})^2} \quad \text{and} \quad \frac{d\varphi}{dz} = \frac{\frac{dz}{d\varphi}}{(\frac{dr}{d\varphi})^2 + (\frac{dz}{d\varphi})^2}$$

and so $v = \frac{K}{(\frac{dr}{d\varphi})^2 + (\frac{dz}{d\varphi})^2}$ from which one finds finally

$$t = \frac{\varepsilon}{K} \int^{\psi_o} \left\{ (\frac{dr}{d\varphi})^2 + (\frac{dz}{d\varphi})^2 \right\} d\varphi \ , \tag{5}$$

the equation of motion along a fixed stream surface ψ_o for axial symmetric three-dimensional flow.

2.3. Solution of the equation of motion.

To solve (5) one must know r and z as functions of φ and ψ and this can be derived at once from equations (3) and (4), owing to the fact that

$$\frac{\psi^2}{Q^r} + (1 - \frac{\psi^2}{Q^r}) = 1 \text{ and } \operatorname{cotgh}^2\alpha - \operatorname{cosech}^2\alpha = 1 :$$

$$(6)\begin{cases} r = l\sqrt{1 - \frac{\psi^2}{Q^2}} \operatorname{cosech} (\frac{2\pi l K\varphi}{Q}) \\ z = -\frac{l\psi}{Q} \operatorname{cotgh} (\frac{2\pi l K\varphi}{Q}) \end{cases}$$

If one puts $\alpha = \frac{2\pi l K\varphi}{Q}$, $\beta = -\frac{\psi}{Q}$, $\beta_o = -\frac{\psi_o}{Q}$

from (6) it follows:

$$\frac{dr}{d\varphi} = -l\sqrt{1 - \beta^2} \operatorname{cosech} \alpha \operatorname{cotgh} \alpha \cdot \frac{2\pi l K}{Q}$$

$$\frac{dz}{d\varphi} = -\beta l \operatorname{cosech}^2\alpha \cdot \frac{2\pi l K}{Q}$$ from which

$$(\frac{dr}{d\varphi})^2 + (\frac{dz}{d\varphi})^2 = \frac{4\pi^2 l^4 K^2}{Q^2} \operatorname{cosech}^2 \alpha (\operatorname{cotgh}^2\alpha - \beta^2)$$

As $d\varphi = \frac{Q}{2\pi lK} d\alpha$ one finds from (5):

$$t = \frac{2\pi\varepsilon l^3}{Q}\left\{\int\frac{1+\sinh^2\alpha}{\sinh^4\alpha}\,d\alpha - \beta_o^2\int\frac{d\alpha}{\sinh^2\alpha}\right\}$$

Now $\int\frac{d\alpha}{\sinh^4\alpha} = -\frac{\cosh\alpha}{3\sinh^3\alpha} - \frac{2}{3}\int\frac{d\alpha}{\sinh^2\alpha}$ (HMF 4.5.86)*

and $\int\frac{d\alpha}{\sinh^2\alpha} = -\operatorname{cotgh}\alpha$

so that $t = \frac{2\pi\varepsilon l^3}{Q}\left\{-\frac{\cosh\alpha}{3\sinh^3\alpha} - \left(-\frac{2}{3} + 1 - \beta_o^2\right)\operatorname{cotgh}\alpha\right\} + C$

which can be evaluated to

$$t = -\frac{2\pi\varepsilon l^3}{3Q}\operatorname{cotgh}\alpha\,(\operatorname{cotgh}^2\alpha - 3\beta_o^2 \quad + C$$

If in general, a particle at t = o is in point P with curvilinear coordinates φ_o and Ψ_o, the equation of motion along the stream surface Ψ_o becomes

$$(7)\quad t = \frac{2\pi\varepsilon l^3}{3Q}\left\{\operatorname{cotgh}\alpha_o\,(\operatorname{cotgh}^2\alpha_o - 3\beta_o^2) - \operatorname{cotgh}\alpha\,(\operatorname{cotgh}^2\alpha - 3\beta_o^2)\right\}$$

with $\alpha = \frac{2\pi lK\varphi}{Q}$ and $\beta = -\frac{\Psi}{Q}$

$\alpha_o = \frac{2\pi lK\varphi_o}{Q}$ and $\beta_o = -\frac{\Psi_o}{Q}$

2.4. Detention times.

From equation (7) one can determine the time an arbitrary particle needs to travel along a stream-surface from one point to another. For instance, if a particle starts at point P (r_o, z_o) and one wants to know the time it needs to reach a point Q at vertical distance z_1 from the origin, one can apply the following procedure:

1° Transform r_o and z_o by means of formulas (1) and (2) into φ_o and Ψ_o.

2° As the particle travels along Ψ_o, r_1 and z_1 must conform with equation (3) with $\Psi = \Psi_o$, from which r_1 can be determined.

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* Handbook of Mathematical Functions, Abramowitz and Stegun, formula 4.5.86

3° Calculate φ_1 by means of formula (1) by substituting r_1 and z_1 for r and z.

4° Determine t by means of equation (7), replacing φ by φ_1.

In this way the detention time of a particle (that is the time a particle needs to reach the pumping well) can be determined. As the theoretical drawdown of the well tends to infinity if r approaches zero, one must put $\varphi = \infty$ in formula (7). Then $\alpha = \infty$ and cotg α = 1 and if then φ_o, Ψ_o, α_o and β_o are replaced by φ, Ψ, α and β, the resulting formula (8) represents a family of surfaces of equal detention times:

$$(8)\quad t_{det} = \frac{2\pi\varepsilon l^3}{3Q}\left\{\text{cotgh}\,\alpha\,(\text{cotgh}^2\alpha - 3\beta^2) + 3\beta^2 - 1\right\}$$

To make a graphical representation of these curves, proceed as follows (see appendix 1):*

1° Construct a dense net of stream- and equipotential-lines by means of the equations (3) and (4).

2° Each point of intersection of these lines has fixed values of φ and Ψ, from which, with help of formula (8), a fixed detention time can be calculated.

3° Connect the points of equal detention time as well as possible.

Along the stream-surface Ψ = o, that is the plane z = o, β = o and thus

$$t_{det} = \frac{2\pi\varepsilon l^3}{3Q}(\text{cotgh}^3\alpha - 1)$$

The value of α for a point P (a,o) can be determined from formula (1):

$$\alpha = \text{arcsinh}\left(\frac{1}{a}\right)$$

from which $\text{cotgh}^2\alpha = \dfrac{a^2+l^2}{l^2}$ and the detention time for P :

$$t_{det}\,(P) = \frac{2\pi\varepsilon}{3Q}\left\{(a^2+l^2)^{3/2} - l^3\right\}$$

This result can be checked by applying the equation of motion

$$t = \varepsilon\int_a^o \frac{dr}{vr}$$

* See remark page 13.

in the plane z = o putting

$$v_r\ (r,o) = K\,\frac{d\varphi}{dr}\,(r,o) = -\frac{Q}{2\pi r}\cdot\frac{1}{\sqrt{r^2+l^2}}$$ from which

$$t = \frac{\pi\varepsilon}{Q}\int_o^a (r^2+l^2)^{1/2}\, d(r^2+l^2) = \frac{2\pi\varepsilon}{3Q}\left\{(a^2+l^2)^{3/2} - l^3\right\}$$

In a similar way the detention time for a point P (0,b) along the z-axis (r=o) can be determined:

$\Psi = -Q \quad \beta = 1 \qquad \alpha = \text{arctgh}\ (\frac{l}{b})$ from which

$\text{cotgh}\,\alpha = \frac{b}{l}$ and

$$t_{det}\ (P) = \frac{2\pi\varepsilon}{3Q}(b^3 - 3bl^2+2l^3)$$

and also $t = \varepsilon\int_b^l \frac{dz}{vz}$ with

$$v_z\ (o,z) = K\,\frac{d\varphi}{dz}\,(o,z) = \frac{Q}{2\pi}\,\frac{1}{l^2-z^2}$$ from which

$$t = \frac{2\pi\varepsilon}{Q}\int_l^b (z^2-l^2)dz = \frac{2\pi\varepsilon}{3Q}(2l^3-3bl^2+b^3).$$

2.5. Moving fronts.

A collection of water particles, initially situated in the plane z=b (b > l) will gradually change in form, as a consequence of the fact that all particles have different velocities.
This change in form with time of such a front can be determined with the aid of the equation of motion along an arbitrary stream-surface (formula 7), by substituting in this formula the equation of the plane z=b in curvilinear coordinates (from formula 6):

$b = \beta_o\ l\,\text{cotgh}\,\alpha_o$ or $\text{cotgh}\,\alpha_o = \frac{b}{\beta_o l}$

and replacing then β_o by β.

The result is:

$$(9)\quad \frac{3Qt}{2\pi\varepsilon} = \frac{b^3}{\beta^3} - 3bl^2\beta - \text{cotgh}\,\alpha\ (\text{cotgh}^2\,\alpha - 3\beta^2)l^3$$

This family of surfaces with parameter t represents the moving front and can be determined graphically in the same way as has been done for formula 8, (see appendix 1)*

In general, if an arbitrary front has the initial form $F(r_o, \Theta_o, z_o) = 0$ in cylinder coordinates (non symmetric), the movement of the front can be determined by replacing r, z, α and β by r_o, z_o, α_o and β_o in formula 6, eliminating from formula 6 and $F(r_o, \Theta_o, z_o) = o$ r_o and z_o, thus expressing α_o as a function of β_o and Θ_o, substituting this in formula 7 and replacing β_o and Θ_o by β and Θ, thus yielding a function $f(\alpha, \beta, \Theta, t) = 0$ or $g(x,y,z,t) = 0$, both representing surfaces of successive positions of the initial front.

One may consider the interface between fresh water and brackish or salt water as such a front, by ignoring the difference in density and assuming the two fluids as immiscible (no dispersion).

The assumption of equal density is allowable if one is dealing with fresh versus brackish water, whereas the neglection of the dispersion phenomenon is more rigorous. The dispersion causes a change of concentration of a tracer along a streamline, this being distinct from the foregoing derivations, where it was assumed that the concentration of a tracer in a fluid particle did not change during the movement of the particle along the stream-surface. In reality there is an interchanging of particles between neighbouring stream-lines (-surfaces) that causes a lateral and transversal spreading of concentration.

In the case of the problem treated here the influence of the dispersion is not so important, for it is not so essential to know that within thirty years or within say thirty years and four months the mixture of pumped fresh and brackish water has an (unallowable) chlorine content.

The third assumption, that of a sharp interface between the fresh and brackish water, is a rather rough approximation. In practice there always occurs a transition zone, where the fresh water gradually changes into brackish and even salt water. These transition zones can be of considerable thickness (10 or 20 m or more) and schematising them to sharp interfaces can influence the results considerably.

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* See remark page 13.

For that reason it is worth trying to make calculations, starting with such an initial distribution of the tracer (the chlorine in our case) that the concentration gradually increases with the depth of the aquifer.

2.6. Initial concentration a function of the depth of the aquifer.

Suppose the initial concentration c_o (at time t = o) is an arbitrary function of z:

$c_o = c_o\,(z)$.

In curvilinear coordinates this equation becomes, applying formula 6:

(10) $c_o^* = c_o^*(\beta l \operatorname{cotgh}\alpha)$

At the time t ($t > o$) a point P (α, β_o) on the stream-surface β_o is occupied by a particle that found itself at rest at t = o in the point P_o (α_o, β_o). The value of α_o can be determined by means of equation 7:

(11) $\operatorname{cotgh}\alpha_o\,(\operatorname{cotgh}^2\alpha_o - 3\beta_o^2) = \dfrac{3Qt}{2\pi\varepsilon l^3} + \operatorname{cotgh}\alpha(\operatorname{cotgh}^2\alpha - 3\beta_o^2)$

The concentration of the tracer in the point P_o at time t = o was

$c_o^* = c_o^*\,(\beta_o\, l \operatorname{cotgh}\alpha_o)$.

This is also the concentration at the time t in the point P. Elimination of cotgh α_o from the latter expression and equation 11 and replacing c_o^* by c^* and β_o by β, yields the function

(12) $F\,(c^*, \alpha, \beta, t) = o$

that is the concentration distribution as a function of time, which can also be written as

$f\,(c, r, z, t) = o$.

The distribution of the concentration of the tracer along the screen of the pumping well as a function of time can be obtained by putting $\varphi = \infty$ or $\alpha = \infty$ in the function (12) which yields the concentration as a function of β and t:

$c = c\,(\beta, t)$

Integrating this function with respect to Ψ from zero to - Q or with respect to β from 0 to 1, gives the mixed concentration c_m of the pumped water as a function of the time only:

(13) $c_m = \int_0^1 c\,(\beta, t)\, d\beta$

In general, the analytical solutions represented by the equations 12 and 13 are only available in implicit form, which implies that numerical results must be obtained in graphical or digital way.
However, with some simplifying assumptions, an explicit solution may sometimes be obtained, as the following example shows.

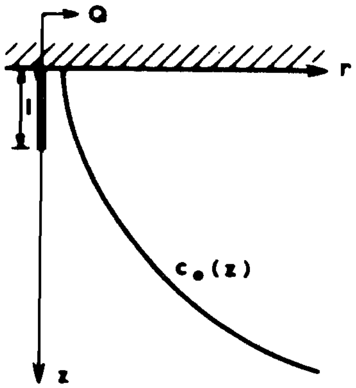


fig. 3

Suppose an initial distribution of the chlorine content such that the concentration increases with the third power of the depth (fig. 3)

(14) $c_o = c_o\ (z) = a + bz^3$

A large value of t after the beginning of the pumping is considered. At that time a particle that was at rest in the point $P_o\ (\alpha_o, \beta_o)$, situated at such a distance from the pumping well that the latter may be replaced by a point source in the origin (which means mathematically that l approaches zero), arives at point $P\ (\alpha,\ \beta_o)$.

As $\lim\limits_{l \to o} (l\,\mathrm{cotgh}\ {}_{o}) = \lim\limits_{l \to o} \left\{ \dfrac{1}{\mathrm{tgh}\left(\frac{2\pi l K\varphi o}{Q}\right)} \right\} = \dfrac{Q}{2\pi K\varphi o}$

equation 11 becomes

$$\left(\frac{Q}{2\pi K\varphi o}\right)^3 = \frac{3Qt}{2\pi\varepsilon} + l^3 \mathrm{cotgh}\,\alpha\ (\mathrm{cotgh}^2\alpha - 3\beta_o^2)$$

and equation (10):

$c_o = a + b\beta_o\ \left(\dfrac{Q}{2\pi K\varphi o}\right)^3$ from which the concentration distribution for large times follows:

(15) $c = a + b\beta\left\{\dfrac{3Qt}{2\pi\varepsilon} + l^3 \mathrm{cotgh}\,\alpha\ (\mathrm{cotgh}\,\alpha^2 - 3\beta^2)\right\}$

The concentration distribution along the well screen ($\alpha = \infty$) becomes

$$c_w = a + b\beta\ \left\{\frac{3Qt}{2\pi\varepsilon} + l^3\ (1 - 3\beta^2)\right\}$$

and the mixed concentration of the pumped water

$c_m = \int_0^1 c_w \, d\beta$ which yields

(16) $c_m = a - \frac{1}{4} bl^3 + \frac{3Qbt}{4\pi \varepsilon}$

As can be seen from the solutions 15 and 16, for large values of time the concentration in a fixed point in the aquifer and also the chlorine content of the pumped water becomes a linear function of the time, if the original concentration is a third power curve with respect to the depth of the aquifer.

Remark: The method to determine lines of equal detention time (page 8) and moving fronts (page 10), mentioned here, is a graphical one. In this case, however, these lines have been drawn by means of a plotter, connected with a desk calculator after programming th- formula's (8) and (9) combined with (1) and (2).

3. Infiltration from a non-penetrating river into an aquifer with thickness D.

As a second example the infiltration of polluted water from a wide river of which the bottom is in connection with a non leaky aquifer of thickness D will be discussed (fig. 4).
Unlike the previous example this problem is two dimensional with a horizontal (x) and a vertical coordinate (y). The infiltration discharge q is maintained by a constant withdrawal of groundwater q or by a fixed drawdown H at a distance $x = L > D$.

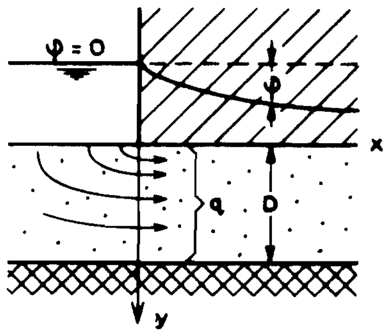


fig. 4

3.1. Analysis of the flow problem.

The quickest way to solve this problem is by means of conformal mapping. This method is based on the fact that in two-dimensional flow the complex function $\Omega = K\varphi + i\Psi$
represents a function of $z = x + iy$:

$$\Omega = \Omega\ (z)$$

From this relation the stream- and potential function can be derived;

$$K\varphi \;=\; \mathrm{Re}\,\Omega = \;K\varphi\ (x,y)$$
$$\Psi \;=\; \mathrm{Im}\,\Omega = \;\Psi\ (x,y).$$

The existence of Ω as a function of z implies that the physical z-plane (fig. 5) is transformed onto the Ω-plane (fig. 6) by means of the function $\Omega = \Omega(z)$ in such a way that the curves φ = constant and Ψ = constant in the z-plane (the potential- and stream-lines) become straight lines in the Ω-plane.

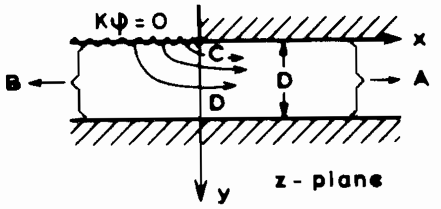


fig. 5

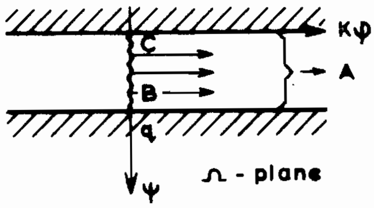


fig. 6

In order to find the function $\Omega(z)$ in a methodical way, both the z- and the Ω-plane are mapped onto the same auxiliary plane $w = u + iv$ (fig. 7) by means of complex functions $z = z(w)$ and $\Omega = \Omega(w)$, such that all boundaries of the flow problem are mapped onto the real u-axis.

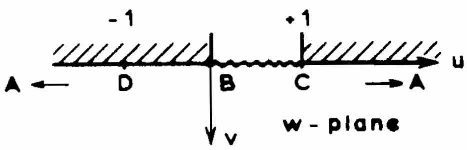


fig. 7

The function $\Omega = \Omega(w)$ and $z = z(w)$ can be determined by means of the Schwarz-Christoffel method.

It follows that:

$z = \frac{D}{\pi} \ln w$ and inversely $w = e^{\frac{\pi z}{D}}$

and $\Omega = \frac{2q}{\pi}$ arccosh w from which

$$\Omega = \frac{2q}{\pi} \operatorname{arccosh}\left(e^{\frac{\pi z}{2D}}\right)$$

with $\Omega = K\varphi + i\Psi$ and $z = x + iy$

3.2. General equation of motion.

In general : $dt = \varepsilon \frac{ds}{v}$ along a streamline

where $v = K\sqrt{\left(\frac{d\varphi}{dx}\right)^2 + \left(\frac{d\varphi}{dy}\right)^2}$

If $\Omega = \Omega\ (z) = K\varphi\ (x,y) + i\psi\ (x,y)$ then

$$\frac{d\Omega}{dz} = K\,\frac{d\varphi}{dx} + i\,\frac{d\psi}{dx} = K\,\frac{d\varphi}{dy} - i\,K\,\frac{d\varphi}{dy}$$

and $\left|\frac{d\Omega}{dz}\right| = K\sqrt{\left(\frac{d\varphi}{dx}\right)^2 + \left(\frac{d\varphi}{dy}\right)^2}$

so that $v = \left|\frac{d\Omega}{dz}\right|$

$ds = \sqrt{(dx)^2 + (dy)^2} = |dz|$ from which

$$dt = \frac{|dz|}{\left|\frac{d\Omega}{dz}\right|} = \left|\frac{dz}{d\Omega}\right|^2 \cdot |d\Omega|$$

$|d\Omega| = \sqrt{(Kd\varphi)^2 + (d\psi)^2}$, but along a streamline = constant, so that $d\psi = 0$

and $|d\Omega| = Kd\varphi$

Hence:

$$t = K\varepsilon \int_{0}^{\psi} \left|\frac{dz}{d\Omega}\right|^2 d\varphi, \tag{18}$$

the general equation of motion along a streamline in two-diemensional steady flow.

3.3. Solution of the equation of motion.

From equation (17) one can derive:

$$z = \frac{2D}{\pi} \ln \cosh \left(\frac{\pi}{2q}\Omega\right) \tag{19}$$

If one puts $\bar{\Omega} = \frac{\pi}{2q}$

then $\frac{dz}{d\Omega} = \frac{2D}{\pi}\,\frac{\sinh\bar{\Omega}}{\cosh\bar{\Omega}} \cdot \frac{\pi}{2q} = \frac{D}{q}\ \mathrm{tgh}\,\bar{\Omega}$ and

$$\left|\frac{dz}{d\Omega}\right|^2 = \frac{D^2}{q^2}\left|\operatorname{tgh}\Omega\right|^2 = \frac{D^2}{q^2}\,\frac{\cosh(2\bar{\varphi}) - \cos(2\bar{\psi})}{\cosh(2\bar{\varphi}) + \cos(2\bar{\psi})}$$

in which $\bar{\varphi} = \frac{\pi K}{2q}\varphi$ and $\bar{\psi} = \frac{\pi}{2q}\psi$

The equation of motion along the streamline ψ_o becomes according to equation 18:

$$t = \frac{\varepsilon D^2}{\pi q}\int^{\psi_o} \frac{\cosh(2\bar{\varphi}) - \cos(2\bar{\psi}_o)}{\cosh(2\bar{\varphi}) + \cos(2\bar{\psi}_o)}\cdot d\,(2\bar{\varphi})$$

$$= \frac{\varepsilon D^2}{\pi q}\left\{\int^{\psi_o} d(2\bar{\varphi}) - 2\cos(2\bar{\psi}_o)\int^{\psi_o}\frac{d(2\bar{\varphi})}{\cosh(2\bar{\varphi}) + \cos(2\bar{\psi}_o)}\right\}$$

Substitute $u = \operatorname{tgh}\bar{\varphi}$

then $\cosh(2\bar{\varphi}) = \frac{1+u^2}{1-u^2}$ and $d(2\bar{\varphi}) = \frac{2du}{1-u^2}$

whence $\int\frac{d(2\bar{\varphi})}{\cosh(2\bar{\varphi}) + \cos(2\bar{\psi}_o)} = \int\frac{2du}{1+u^2 + (1-u^2)\cos(2\bar{\psi}_o)}$

$$= \int\frac{2du}{1 + \cos(2\bar{\psi}_o) + \left\{1 - \cos(2\bar{\psi}_o)\right\}u^2} = \int\frac{du}{\cos^2\bar{\psi}_o + u^2\sin^2\bar{\psi}_o}$$

$$= \frac{1}{\sin\bar{\psi}_o\cos\bar{\psi}_o}\operatorname{arctg}\left(u\,\frac{\sin\bar{\psi}_o}{\cos\bar{\psi}_o}\right)$$

$$= \frac{2}{\sin(2\bar{\psi}_o)}\operatorname{arctg}(\operatorname{tg}\bar{\psi}_o\operatorname{tgh}\bar{\varphi})$$ and thus

$$t = \frac{2\varepsilon D^2}{\pi q}\left\{\bar{\varphi} - 2\operatorname{cotg}(2\bar{\psi}_o)\operatorname{arctg}(\operatorname{tg}\bar{\psi}_o\operatorname{tgh}\bar{\varphi})\right\} + c \qquad (20)$$

3.4. Infiltration of polluted water.

Now the bottom of the river is considered as the initial front of a waterbody of in some way polluted water, that is at $t = o$, the equation of the front is

$z = o \quad x \leq o$ or $\varphi = o$

and also $\bar{\varphi} = o$

Equation 20 then becomes with $\bar{\Psi}_o = \bar{\Psi}$

$$(21)\quad \frac{\pi q t}{2\varepsilon D^2} = \bar{\varphi} - 2\ \mathrm{cotg}\ (2\bar{\Psi})\mathrm{arctg}\ (\mathrm{tg}\ \bar{\Psi}\ \mathrm{tgh}\ \bar{\varphi}) = \bar{t}$$

This formula represents the movement of the front as a function of time in curvilinear coordinates (see appendix 2). Transition to Cartesian coordinates can be performed by means of the relation (17) which can be evaluated to:

$$(22)\quad \begin{aligned} \bar{\varphi} &= \mathrm{arccosh} \left\{ \frac{1}{2}\sqrt{v^2+(u+1)^2} + \frac{1}{2}\sqrt{v^2+(u-1)^2} \right\} \text{ and} \\ \bar{\Psi} &= \mathrm{arccos} \left\{ \frac{1}{2}\sqrt{v^2+(u+1)^2} - \frac{1}{2}\sqrt{v^2+(u-1)^2} \right\} \end{aligned}$$

with $\mu = 2^{\frac{\pi x}{2D}} \cos(\frac{y}{2D})$ and $v = e^{\frac{\pi x}{2D}} \sin(\frac{\pi y}{2D})$.

It can be shown that at a distance $x = L > D$ the flow becomes virtually horizontal.
If a row pumping wells parallel to the river, at distance L (>D), pumpes with a capacity q, the average drawdown φ at $x = L$ can be found by calculating $\bar{\varphi}$ (L,o) from formula 22:

$$u(L,o) = e^{\frac{\pi L}{2D}},\quad v(L,o) = o$$

$$\bar{\varphi}(L,o) = \mathrm{arccosh}(e^{\frac{\pi L}{2D}}) = \bar{\varphi}(L)$$

Substituting this value in equation 21 yields the detention time of the polluted water in the soil, that is the time a particle needs to travel from the river to the row of wells along a fixed streamline.

$$\cosh \bar{\varphi}(L) = e^{\frac{\pi L}{2D}} \text{ from which } \mathrm{tgh}\ \bar{\varphi}(L) = \sqrt{1-e^{-\frac{\pi L}{D}}} \quad \text{and so}$$

$$(23)\quad \bar{t}_{det} = \mathrm{arccosh}\ (e^{\frac{\pi L}{2D}}) - 2\ \mathrm{cotg}\ (2\bar{\Psi}_o)\ \mathrm{arctg}\ (\mathrm{tg}\ \bar{\Psi}_o \sqrt{1 - e^{-\frac{\pi L}{2D}}})$$

A particle, travelling along the bottom of the impermeable layer, that is along the streamline $\bar{\Psi} = o$, has a minimum detention time. This minimum detention time is also the critical time, that is the time at which the first polluted water particle reaches the row of wells. Putting $\bar{\Psi}_o = o$ in equation 23 and applying the rule of l'Hôpital one obtains for the critical time:

$$(24)\quad \bar{t}_{cr} = \operatorname{arccosh}\left(e^{\frac{\pi L}{2D}}\right) - \sqrt{1-e^{-\frac{\pi L}{2D}}}$$

If the water of the river contains a certain concentration c of dissolved solids, then the concentration c_m of the pumped water can be determined as a function of the time, as follows:

For $\bar{t} \leq \bar{t}_{cr}$, $c_m = o$

At the time $\bar{t}$ ($\geq \bar{t}_{cr}$) a certain part of the filter screen in vertical direction attracts water with a concentration c where as the other part pumpes water with a concentration zero. The two zones are separated from each other by the intersection of the screen with streamline $\bar{\Psi}_o$, where $\bar{\Psi}_o$ follows from formula 23.

As $\bar{\Psi}_o$ runs from o to $\frac{\pi}{2}$, the mixed concentration of the pumped water becomes

$$c_m = \frac{\bar{\Psi}_o}{\pi/2}\, c$$

Elimination of $\bar{\Psi}_o$ from this expression and formula 23 yields the mixed concentration of the pumped water as a function of the time:

$$(25)\quad \bar{t} = \operatorname{arccosh}\left(e^{\frac{\pi L}{2D}}\right) - 2\operatorname{cotg}\left(\frac{\pi c_m}{c}\right)\operatorname{arctg}\left\{\operatorname{tg}\left(\frac{\pi c_m}{2c}\right)\sqrt{1-e^{-\frac{\pi L}{2D}}}\right\}$$

In appendix 3 $\frac{c_m}{c}$ has been drawn as a function of $\bar{t}$ for some fixed values of $\frac{L}{D}$.

PARTIALLY PENETRATING WELL IN A DEEP AQUIFER
DETENTION TIMES AND UPCONING OF A HORIZONTAL FRONT

$\alpha = \frac{2\pi K l \varphi}{Q}$

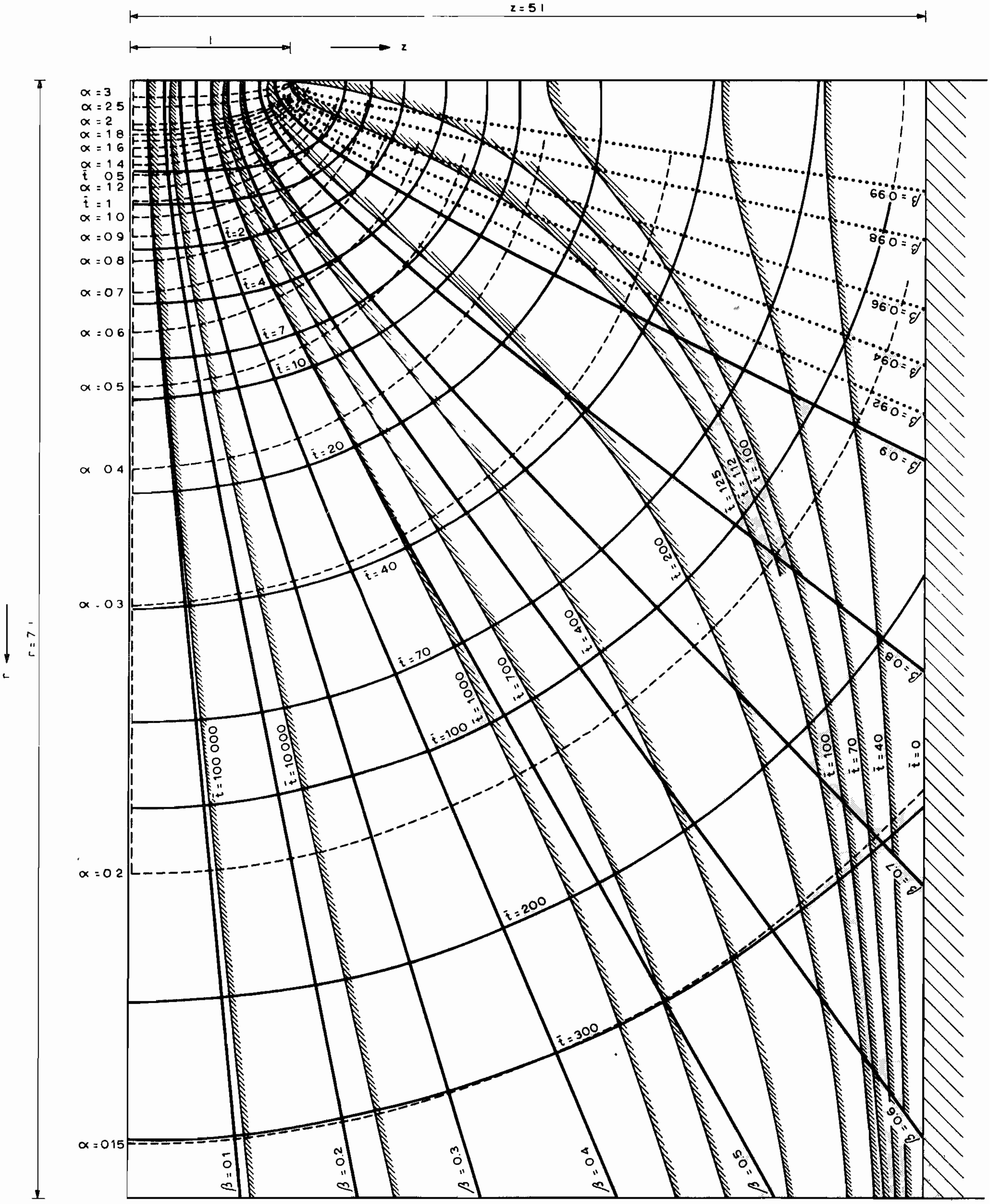
$\beta = -\frac{\Psi}{Q}$

$\bar{t} = \frac{3Qt}{2\pi \varepsilon l^3}$

Streamlines (β)

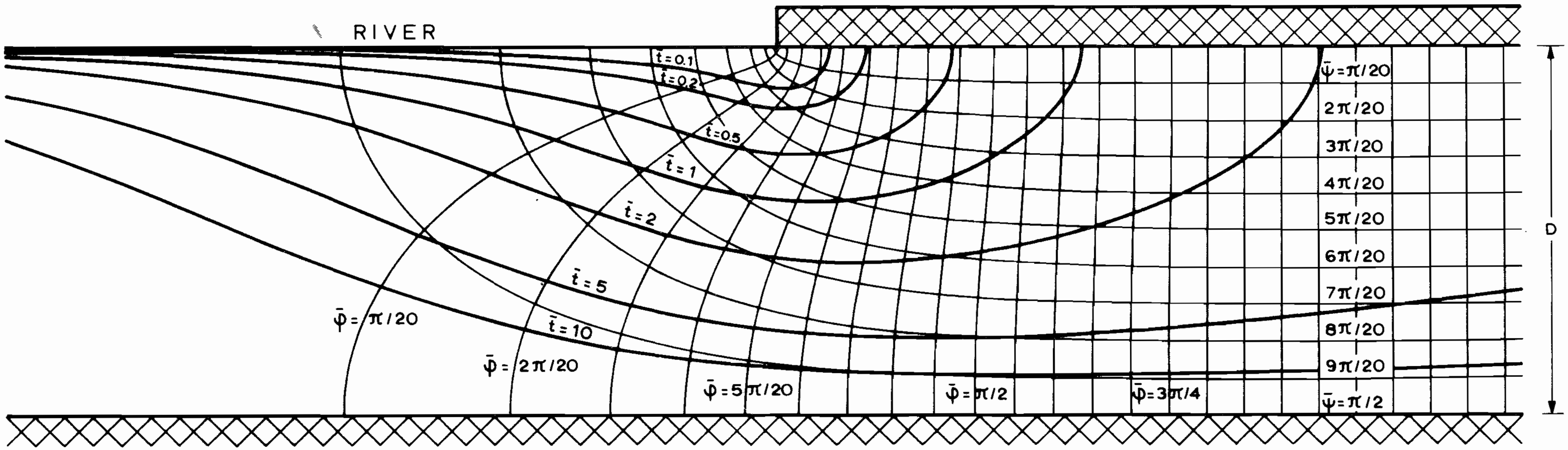
Aequipotential lines (α)

Moving fronts ($\bar{t}$)



INFILTRATION FROM A NON-PENETRATING RIVER INTO AN AQUIFER WITH THICKNESS D

$$\bar{t} = \frac{\pi q t}{2 \varepsilon D^2}$$



<u>Practical example</u>

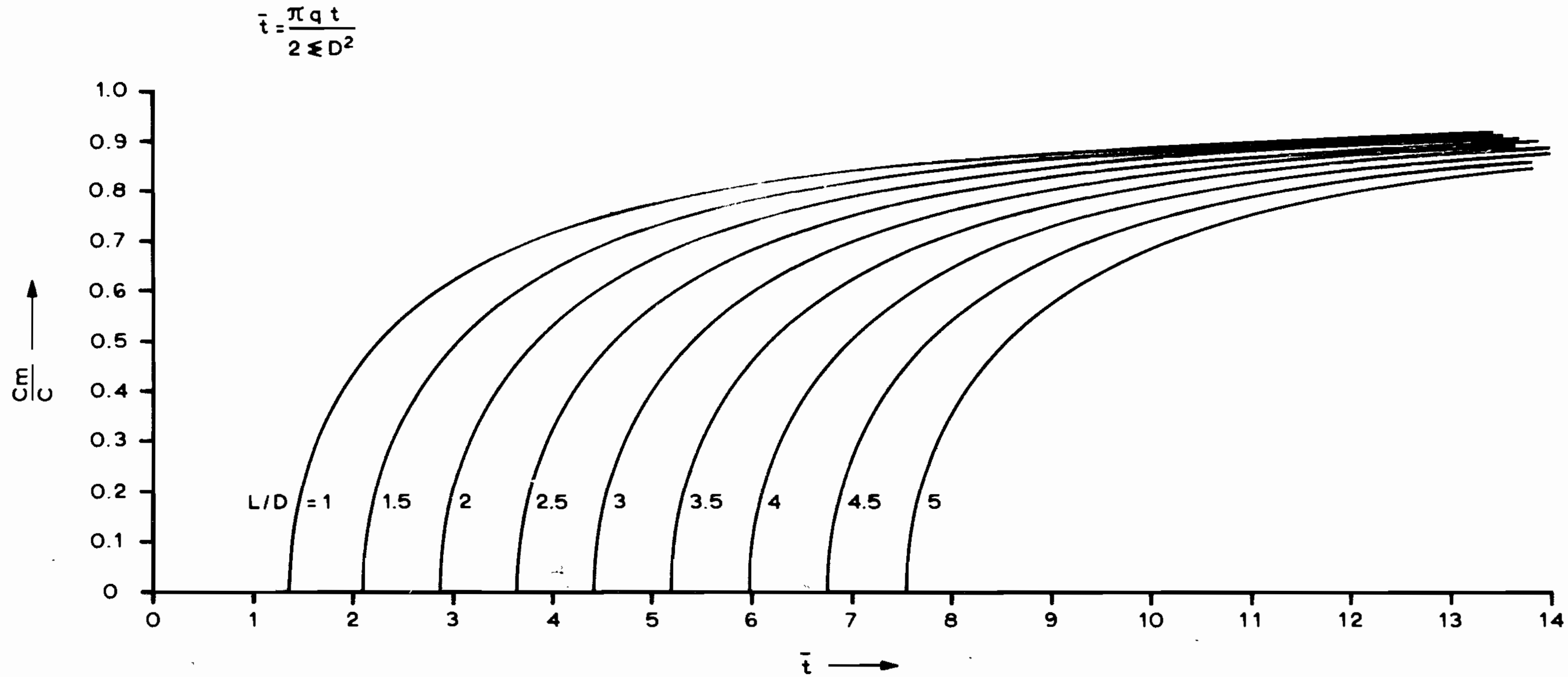
$q = 5$ m /day

$\varepsilon = 0.25$

$D = 25$ m

Then $\bar{t} = 1/20\,t$

CONCENTRATION OF DISSOLVED SOLIDS AS A FUNCTION OF TIME IN WATER, PUMPED NEAR A NON-PENETRATING RIVER



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