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Mathematical analysis of steady- and non-steady two- and three-dimensional flow in anisotropic aquifers

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This Communication no 73-1 of the Government Institute for Water Supply comprises the author's contribution to the Salt-Water Intrusion Meeting, held at Copenhagen, from 6-9 June 1972.

As the subject of that paper was not exclusively concerned with fresh-salt water problems but had a much wider scope, dealing with general methods in obtaining analytical solutions for more dimensional steady and unsteady groundwater problems, it was thought useful to give it a somewhat wider propagation by publishing it as a Communication of the Institute, with a slightly modified title.

MATHEMATICAL ANALYSIS OF STEADY AND NON-STEADY TWO- AND THREE-DIMENSIONAL FLOW IN ANISOTROPIC AQUIFERS WITH SPECIAL REFERENCE TO SALT WATER INTRUSION PROBLEMS.

1. Introduction

In this paper not so much stress will be laid upon salt water intrusion problems as well upon several methods to obtain analytical solutions of ground water flow problems in general, which however, will be of particular interest for the analytical approach of salt water intrusion problems.

These problems always arise from changes in the existing groundwater flow system, mostly caused by human activities, like abstraction of groundwater by means of pumping wells, digging building pits, land reclamation, etc.

For instance, if the shape of an upconing salt-fresh water interface caused by a groundwater withdrawal must be predicted to a certain degree of exactness, a good knowledge of the stream- and potential lines governing the flow problem is necessary, whether phenomena like dispersion, or the difference in density between fresh and salt or brackish water are ignored or not. If dispersion is taken into account at first the groundwater velocity components must be determined as functions of place and sometimes of time before they are introduced in the differential equation that governs the dispersion phenomenon. If both the dispersion and the difference in density are neglected (the brackish groundwater flow thus considered as an ideal immiscible two-liquid flow) the assumed sharp interface between fresh and brackish water must be determined as a function of time and position. In all cases however at first a solution of the groundwater flow apart is required. Thus an important rule is:

The basis of dealing with salt water intrusion problems is a good knowledge of the methods for solution of groundwater flow problems in general.

This seems to be self-evident. Nevertheless it must be stated that many geohydrologists are quite unfamiliar with the mathematical approach of groundwater problems with more than one dimension.

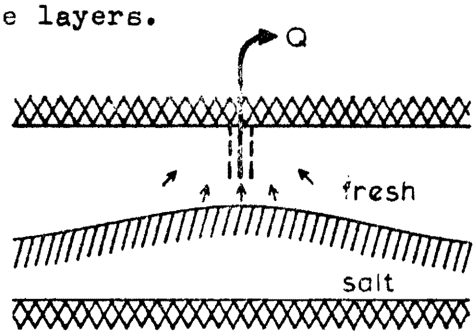
Indeed, most problems concerning groundwater flow are dealt with in a one-dimensional way; that means that only horizontal flow in the aquifer towards wells, drains, canals, reservoirs etc. is considered and if possible (and even if it is impossible) they are treated as stationary problems.

This is in a way understandable, because, if the problem is treated one-dimensionally and stationary it is governed by a one-dimensional or ordinary differential equation, which in most cases can be solved easily and in fact in many geohydrological problems, when a steady state presents it self and where for instance the magnitude of the seepage through a dam or the discharge of a series of pumping wells are the unknowns to be solved, the neglection of the vertical flow is acceptable. Even in problems of partial penetrating wells, canals etc., difficulties can be avoided by application of the rule that at a distance of

$\sim 1{,}5\, D\sqrt{\frac{K_v}{K_h}}$ the effect of the partial penetration will have vanished,

in which D = thickness of the aquifer and K_v and K_h are the coeffcients of vertical and horizontal permeability respectively.

However, as soon as groundwater flow computations must be performed in order to get a better insight into the attracting and especially into the raising of deeper groundwater with a higher Cl-content, either the vertical components of the groundwater velocity must be taken into account, when dealing with one aquifer, or, in case of more aquifers separated by semipermeable layers, the problem must be treated as a more-layer system with horizontal flow in the aquifers and vertical flow through the semi-permeable layers.



Thus one arrives at a second basic rule:
Salt water intrusion problems in groundwater flow can only be treated adequately by means of two- or threedimensional analysis with even an extra dimension, the time, in non-steady cases.

The main consequence of this rule is that, if exact solutions are wanted, methods must be available to solve partial differential equations, or in the case of more-layer systems, to solve simultaneous differential equations.

Of course in most cases numerical solutions can be found or a solution by means of an analogon and in fact for complicated problems only these last two devices may lead to an acceptable result, but exact solutions if they, at least, can be obtained in such a way that they are fit for practical use, are always preferable.

A second consequence of the given rule is that if vertical flow presents itself, also the phenomenon anisotropy may play a role. Anisotropy of the underground means that unlike isotropic ground, the permeability coefficient of the ground varies with the direction, in a homogeneous anisotropic aquifer this variation being the same in every point of the aquifer.

In practice, anisotropy occurs more frequently than isotropy; especially the difference in vertical and horizontal permeability in sediments plays an important role. This difference can be considerable and a horizontal permeability of over 50 times the vertical permeability is not unusual. For this reason the following rule holds:

<u>With regard to sedimental layers with salt water intrusion problems the anisotropy of the underground always should be taken into account.</u>

If anisotropy is neglected and the horizontal permeability is chosen to represent the whole aquifer, the results of the calculation of for instance an upconing of a salt water body will be too unfavourable, as the calculated vertical groundwater velocities will turn out to be too high in comparison with the real velocities.

In analytical computations the anisotropy does not cause much trouble as far as the directions of the main permeabilities coincide with the directions of the coordinate axes, as in most problems will be the case. Then by means of a skilful substitution the difference in the permeabilities can be eliminated in the differential equation without affecting its character.

For example, consider the differential equation for two-dimensional non-steady flow:

$$K_h \frac{\partial^2 \varphi}{\partial x^2} + K_v \frac{\partial^2 \varphi}{\partial z^2} = S_s \frac{\partial \varphi}{\partial t}$$

Divide by K_h and put $\frac{K_v}{K_h} = a^2$

$$\frac{\partial^2 \varphi}{\partial x^2} + a^2 \frac{\partial^2 \varphi}{\partial z^2} = \frac{S_s}{K_h} \cdot \frac{\partial \varphi}{\partial t}$$

Substitute $z = az_1$,

then $\frac{\partial^2 \varphi}{\partial z^2} = \frac{1}{a^2} \frac{\partial^2 \varphi}{\partial z_1^2}$

and the differential equation becomes:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z_1^2} = \frac{S_s}{K_h} \frac{\partial \varphi}{\partial t}$$

and this is again the "normal" differential equation for isotropic ground with one permeability (the horizontal one) for all directions: the groundwater body has been extracted figuratively in the z-direction with a factor $\frac{1}{a}$ (> 1 if $K_v < K_h$)

In this connection it must be pointed out that in anisotropic ground the stream- and aequipotential lines are <u>not perpendicular</u> to each other (except at points where a streamline has the direction of one of the main permeabilities).

By means of special non-steady pumping tests with partial penetrating pumping wells and observation wells at several depths, the values of the horizontal as well as of the vertical permeability may be determined. A method for this determination has been worked out at our Institute and has been applied with succes.

2. Differential equations and calculation techniques.

The general differential equation for three-dimensional non-steady groundwater flow in the saturated zone in homogeneous, isotropic ground, runs as follows:

$$\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2} + F\,(x,y,z,t) = \frac{S_s}{K}\ \frac{d\varphi}{dt}$$

in which

φ = the potential head of the groundwater

x, y, z = place variables

t = time variable

S_s = specific storage coefficient

F (x,y,z,t) = a function that denotes the way in which water is generated or abstracted into or from the groundwater body under consideration.

In dealing with steady flow the term with $\frac{d\varphi}{dt}$ and without injection or abstraction also the term F can be deleted. In that case the differential equation for steady flow through homogeneous isotropic ground reduces to the three-dimensional differential equation of Laplace:

$$\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2} = 0$$

If there is axial symmetry, the use of cylinder coordinates is obvious; the differential equation then becomes:

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} + \frac{d^2\varphi}{dz^2} = 0$$

The greater number of salt water intrusion problems will be governed by the two following differential equations and in this paper we shall restrict ourselves to these:

1e $\quad \dfrac{d^2\varphi}{dx^2} + \dfrac{d^2\varphi}{dz^2} = \dfrac{S_s}{K}\dfrac{d\varphi}{dt}$ (two-dimensional flow)

2c $\quad \dfrac{d^2\varphi}{dr^2} + \dfrac{1}{r}\dfrac{d\varphi}{dr} + \dfrac{d^2\varphi}{dz^2} = \dfrac{S_s}{K}\dfrac{d\varphi}{dt}$ (three dimensional axial symmetric flow)

and to the analogue equations for steady flow (without $\frac{d\varphi}{dt}$).

All here mentioned equations are partial differential equations and the main problem with them is that unlike ordinary differential equations a general solution with some constants cannot be found. A partial differential equation has to be solved together with the initial- and boundary values. This may be the reason that many geohydrologists do not know to deal with them. However, there are adequate methods to solve partial differential equations and some of them will be the subject of this paper.

a. Conformal mapping

In the first place the very elegant method of conformal mapping must be mentioned. This method may be used if the problem is a so-called potential flow problem, governed by the differential equation:

$$\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dz^2} = 0 \qquad \text{or} \qquad \frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} = 0 \quad ,$$

the last one not being of interest for upconing problems (only horizontal flow in two directions).

In many handbooks on hydrology this method is treated and for the time being it will suffice to refer to these books (for a thorough treatment the "Theory of Groundwater Movement" by

Mrs. Polibarinova-Kochina can be recommended) and to draw attention to the fact that almost every potential flow problem in homogeneous confined aquifers with a constant thickness and straight boundaries and also a number of problems concerning phreatic water or a salt-fresh water interface can be solved exactly by means of this complex plane method and especially the Schwarz-Christoffel and hodograph version of it.

b. Integral transformations

A disadvantage of the method of conformal mapping is that only steady two-dimensional problems can be solved and as soon as time or a third dimension or seepage through a semi-permeable layer present themselves the method cannot be used and other methods have to be applied. Among them the integral transformation methods are very powerful tools for obtaining exact solutions, but they are surprisingly little known or at least scarcely applied in geohydrology.

The integral transformation techniques are based on the fact, that by means of a skilful substitution in integral form one of the dimensions of the differential equation can be eliminated and by continuing this procedure the partial differential equation can be reduced to an ordinary differential equation or even to a common algebraic equation, which usually will not yield any difficulties for their solution with help of the likewise transformed boundary values.

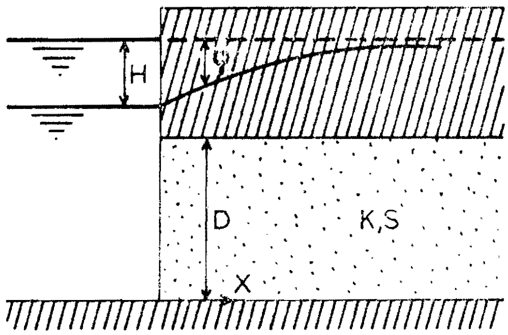
The in this way obtained solution must undergo a reverse or some reverse transformations in order to obtain the desired solution of the problem.

b.1. Laplace transformation

The Laplace transformation generally is used to eliminate the time variable; so it is most frequently applied to non-steady problems. The following example will show us the procedure.

Consider a confined aquifer of thickness D, infinitely extended in y-direction and semi-infinitely in x-direction ($o < x < \infty$) while the aquifer at x=o stands in open connection with an infinite, fully penetrating canal.
By means of a sudden lowering of the canal level, which further is kept constant a drawdown of the piezometric head, dependant of time and place, is introduced. The initial drawdown is assumed to be zero.

The problem is non-steady geographically one-dimensional (only x-coordinate).



The drawdown φ must be determined as a function of x and t:
$\varphi = \varphi\ (x,t)$.
The boundary value problem becomes.

$$\frac{d^2\varphi}{dx^2} = \frac{S}{KD} \cdot \frac{d\varphi}{dt}$$

$\varphi(x,o)=o$, $\varphi(o,t)=H$, $\varphi\ (\infty,t)=o$

in which K and S represent de permeability- and the storage coefficient of the aquifer respectivily.
Now the unknown function $\varphi\ (x,t)$ is multiplied by the factor e^{-st} and integrated with respect to t from zero to infinity, thus obtaining a new function, independant of t:

$$L\left\{\varphi(x,t)\right\} = \int_o^\infty e^{-st} \cdot \varphi(x,t)dt = \bar{\varphi}(x,s) \quad \text{(s an arbitrary positive value)}$$

This operation on the function $\varphi(x,t)$ is called the Laplace transformation of $\varphi(x,t)$ and the new function is called the Laplace transform, which will usually be indicated by L $\left\{\varphi\right\}$ or by means of a bar: $\bar{\varphi}\ (x,s)$ or shortly $\bar{\varphi}$.
The same operation applying to the differential equation, gives:

$$L\left\{\frac{d\varphi}{dt}\right\} = \int_o^\infty \frac{d\varphi}{dt} \cdot e^{-st}dt = \int_o^\infty e^{-st}\ d\varphi = \left[\varphi e^{-st}\right]_o^\infty - \int_o^\infty \varphi\ d(e^{-st}) =$$

$$- \varphi(x,o) + s\int_o^\infty e^{-st}\ \varphi(x,t)dt = s\bar{\varphi} - \varphi(x,o)$$

$$\text{and } L\left\{\frac{d^2\varphi}{dx^2}\right\} = \int_o^\infty \frac{d^2\varphi}{dx^2}\ e^{-st}\ dt = \frac{d^2}{dx^2}\int_o^\infty \varphi e^{-st}dt = \frac{d^2\bar{\varphi}}{dx^2} \ .$$

The transformed differential equation becomes:

$$\frac{d^2\bar{\varphi}}{dx^2} = \beta^2\, s\bar{\varphi} - \beta^2\bar{\varphi}(x,o) \quad \text{and as } \varphi(x,o) = o \text{ (initial value):}$$

$$\frac{d^2\bar{\varphi}}{dx^2} - \beta^2 s\bar{\varphi} = o \quad \text{with} \quad \beta^2 = \frac{S}{KD}\,.$$

This differential equation is an ordinary one, with only one variable x and the device of the transformation is obvious: to eliminate the differential quotient $\frac{d\varphi}{dt}$ and to reduce it to the transformed function $\bar{\varphi}$. A further advantage of the transformation is that the initial value at the same time is incorporated.

Now the two remaining boundary values still have to be transformed:

$$\bar{\varphi}\ (\infty,s) = o \text{ and } \bar{\varphi}\ (o,s) = \int_o^\infty H.e^{-st}dt = \left[-\frac{H}{s}\,e^{-st}\right]_o^\infty = \frac{H}{s}$$

The transformed boundary value problem thus becomes:

$$\left[\frac{d^2\bar{\varphi}}{dx^2} - \beta^2\, s\bar{\varphi} = o \quad , \quad \bar{\varphi}\ (\infty,s) = o \quad , \quad \bar{\varphi}\ (o,s) = \frac{H}{s}\right]$$

with the transformed solution: $\bar{\varphi}(x,s) = \frac{H}{s}\, e^{-\beta x\sqrt{s}}$

From tables of Laplace transforms the reverse transform can be found as:

$\varphi(x,t) = H \text{ erfc } \left(\frac{\beta x}{2\sqrt{t}}\right) = H \text{ erfc } \left(\frac{x}{2}\sqrt{\frac{S}{KDT}}\right)$ and this is the desired exact solution, in which erfc(z) = complementary error function = $1 - \text{erf}(z) = 1 - \frac{2}{\sqrt{\pi}}\int_o^z e^{-\lambda^2}d\lambda$ and indeed, the Laplace transform of this solution:

$$H\int_o^\infty e^{-st} \text{ erfc } \left(\frac{\beta x}{2\sqrt{t}}\right) dt:$$

can be evaluated to $\quad \frac{H}{s}\, e^{-\beta x\sqrt{s}}$

Analogue problems with varying water table in the canal and even variation according to an arbitrary function of the time F (t) can be solved easily by means of the Laplace transformation technique. Besides this, numerous other examples could have been given, but from this simple example the importance of the Laplace transformation method for solution of non-steady problems will have become clear.

b.2. Hankel transformations

The Hankel transformations can be divided into finite and infinite Hankel transformations. Both transformations are applied to axial symmetric problems, that is in all problems where the independant variable r plays a role.

By the infinite Hankel transformation the operation is understood in which the unknown function φ (r) is multiplied by the factor r J_o (αr) and the product is integrated with respect to r from zero to infinity, thus obtaining a new function, denoted as $H\{\varphi(r)\}$ or as $\hat{\varphi}(\alpha)$, independant of r:

$$\hat{\varphi}(\alpha)=\int_0^\infty r\ \varphi(r)\ J_o\ (\alpha r)\ dr$$

in which $J_o(\alpha r)$ is the Besselfunction of the first kind and of zero order and α has an arbitrary positive value.
For instance the transform of $f(r) = \frac{1}{r}\, e^{-cr}$ becomes:

$$\hat{f}(\alpha) = \int_0^\infty \frac{1}{r}\, e^{-cr}.\ r\ J_o(\alpha r)dr = \int_0^\infty e^{-cr}\ J_o(\alpha r)dr = \frac{1}{\sqrt{\alpha^2+c^2}}$$

(Laplace integral)

The main property from which the infinite Hankel transformation derives its value for solving axial symmetric problems is, that it reduces the terms

$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr}$, which always occur in the differential equations of those problems, to the transformed function itself, as can be shown as follows:

$$\int_0^\infty \left(\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr}\right) r\ J_o(\alpha r)dr = \int_0^\infty \left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right)\right\}\ r\ J_o(\alpha r)dr =$$

$$\int_0^\infty J_o(\alpha r)\ d\ \left(r\frac{d\varphi}{dr}\right) = J_o(\alpha r)\ r\frac{d\varphi}{dr} - \int_0^\infty r\frac{d\varphi}{dr}\ d\left\{J_o(\alpha r)\right\} =$$

$$J_o(\alpha r)\, r\,\frac{d\varphi}{dr} + \alpha\int_o^\infty r\, J_1(\alpha r)\, d\varphi = J_o(\alpha r)\, r\,\frac{d\varphi}{dr} + \alpha r\, J_1(\alpha r)\,.\,\varphi$$

$$-\,\alpha\int_o^\infty \varphi\, d\left\{r\, J_1(\alpha r)\right\} = r\, J_o(\alpha r)\,\frac{d\varphi}{dr} + \alpha r\, J_1(\alpha r)\,\varphi - \alpha^2\int_o^\infty \varphi\, r\, J_o(\alpha r)\, dr =$$

$$\left[r\, J_o(\alpha r)\,\frac{d\varphi}{dr} + \alpha r\, J_1(\alpha r)\,\varphi\right]_o^\infty - \alpha^2\,\hat{\varphi}\,(\alpha),$$ in which $J_1(\alpha r)$=Besselfunction of the first kind and first order.

So, if the function $\varphi(r)$ satisfies the condition that both φ and $\frac{d\varphi}{dr}$ vanishes for $r = \infty$, we obtain:

$$H\left\{\frac{d^2\varphi}{dr^2} + \frac{1}{r}\,\frac{d\varphi}{dr}\right\} = -\lim_{r\to o}\left(r\,\frac{d\varphi}{dr}\right) - \alpha^2\,\hat{\varphi}\,(\alpha)$$

<u>Hence, application of the infinite Hankel transformation is possible for an axial symmetric problem for which the horizontal velocity at $r = o$ is given and the potential drawdown or elevation and the groundwater velocity at infinity can be assumed zero.</u>

From the theory of Bessel functions it is known that an arbitrary function $f(r)$ under certain conditions can be represented by the Hankelintegral, as follows:

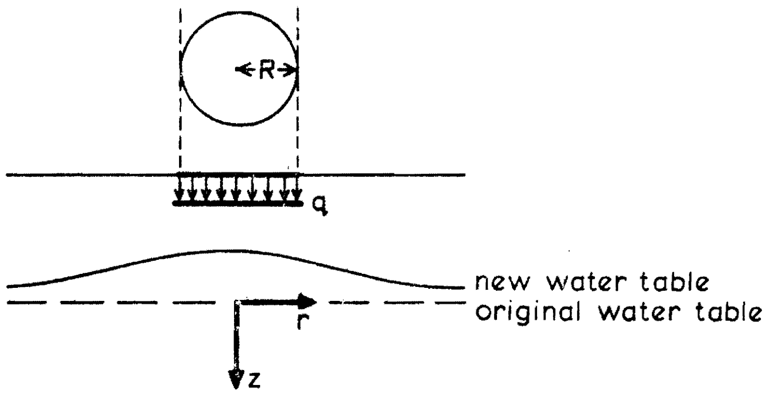
$$f(r) = \int_o^\infty \alpha\, A(\alpha)\, J_o(r\alpha)\, d\alpha \quad \text{in which}$$

$$A(\alpha) = \int_o^\infty r\, f(r)\, J_o(\alpha r)\, dr$$

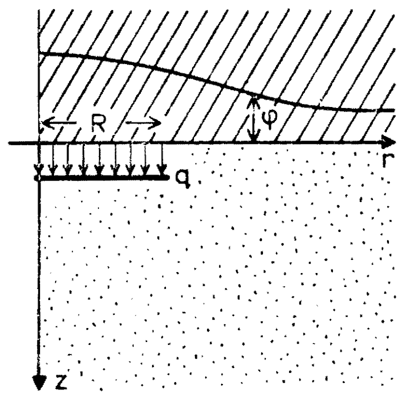
As $A(\alpha)$ is the Hankel transform of $f(r)$ or $A(\alpha) = \hat{f}(\alpha)$ the inverse Hankel transformation turns out to be very simple:

$$\varphi(r) = \int_o^\infty \alpha\,\hat{\varphi}(\alpha)\, J_o(r\alpha)\, d\alpha\,.$$

As an example consider infiltration with a constant velocity q from a circular pond into an assumed infinitely thick aquifer, which will yield a steady curved phreatic surface.



For simplification the water table aquifer may be approximated to a confined aquifer in such a way that the original horizontal water table coincides with the bottom of the impermeable layer that covers the aquifer, while water supply takes place at the rate q per unit time per unit area over a circular disc. The boundary conditions of the curved unknown phreatic surface are thus reduced to boundary conditions in a fixed plane. Now the elevation φ of the original piezometric head has to be determined as a function of r and z: $\varphi = \varphi(r,z)$.



The boundary value problem can be translated mathematically as follows:

$$\left[\begin{array}{l} \dfrac{d^2\varphi}{dr^2} + \dfrac{1}{r}\dfrac{d\varphi}{dr} + \dfrac{d^2\varphi}{dz^2} = o \\[2ex] \varphi\,(\infty, z) = o \qquad \dfrac{d\varphi}{dr}\,(o, z) = o \\[2ex] \varphi\,(r, \infty) = c \qquad \dfrac{d\varphi}{dz}\,(r, o) = -\dfrac{q}{K} \quad \text{for } o < r < R \\ \qquad\qquad\qquad\qquad\qquad\quad = \quad o \quad \text{for } r > R \end{array}\right]$$

Hankel transformation with respect to r of the discontinuous boundary condition for z = o yields:

$$\frac{d\hat{\varphi}}{dz}\,(\alpha, o) = \int_o^\infty \frac{d\varphi}{dz}\,(r, o)\; r\; J_o(\alpha r)\; dr = -\frac{q}{K}\int_o^R r\; J_o(\alpha r)\; dr =$$

$-\frac{qR}{K\alpha} J_1\ (\alpha R)$ whereupon the transformed boundary value problem becomes:

$$\left[\frac{d^2\hat{\varphi}}{dz^2} - \alpha^2\hat{\varphi} = o\ ,\quad \hat{\varphi}\,(\alpha, \infty) = o\ ,\quad \frac{d\hat{\varphi}}{dz}\,(\alpha, o) = -\frac{qR}{K\alpha} J_1\ (\alpha R)\right]$$

The solution of this ordinary differential equation is:

$$\hat{\varphi}\,(\alpha, z) = \frac{qR}{K\alpha^2} J_1\ (\alpha R) e^{-\alpha z}$$

Inverse infinite Hankel transformation gives the desired solution:

$$\varphi(r, z) = \frac{qR}{K}\int_o^\infty \frac{1}{\alpha} J_1\ (R\alpha)\ J_c\ (r\alpha) e^{-z\alpha}\ d\alpha.$$

which expression can be evaluated in infinite series and thus calculated for every value of r and z. Along the coordinate axes the integral function reduces to transcendental functions, for example along the z-axes:

$$\varphi(o,z) = \frac{qR}{K}\int_0^\infty \frac{1}{\alpha} J_1(R\alpha)e^{-z\alpha}\, d\alpha = \frac{q}{K}\left(\sqrt{R^2+z^2} - z\right)$$

(Laplace integral)

and along the r-as:

$$\varphi(r,o) = \frac{qR}{K}\int_0^\infty \frac{1}{\alpha} J_1(R\alpha)\, J_o(r\alpha)\, d\alpha \qquad \text{(Weber-Schaftheitlin integral)}$$

$$= \frac{2qr}{\pi K} E\left(\frac{r^2}{R^2}\right) \text{ for } r < R \qquad \text{and}$$

$$= \frac{2q\,(R^2-r^2)}{\pi K r} K\left(\frac{R^2}{r^2}\right) + \frac{2qr}{\pi K} E\left(\frac{R^2}{r^2}\right) \text{ for } R > r$$

in which $K(z)$ and $E(z)$ represent complete elliptic integrals of the first and second kinds respectively.

From the theory of Bessel functions it is known that under certain conditions an arbitrary function can be represented by a so called Fourier-Bessel series:

$$f(x) = \sum_{n=1}^{\infty} c_n J_o\left(\frac{\alpha_n x}{a}\right) = c_1 J_o\left(\frac{\alpha_1 x}{a}\right) + c_2 J_o\left(\frac{\alpha_2 x}{a}\right) + \ldots\ldots \text{ ad inf.}$$

$$\text{with } c_n = \frac{2}{a^2 J_1^{\,2}(\alpha_n)} \int_0^a x f(x) J_o\left(\frac{\alpha_n x}{a}\right) dx$$

and in which α_n $(n=1,2\ \ldots)$ are the roots of the equation $J_o(\alpha) = o$

Now we can consider the integral in this expression as an operation performed on the function $f(x)$ and we call this operation the <u>finite Hankel transformation</u>, for instance:

$$H_n\left\{\varphi(r)\right\} = \int_0^R r\varphi(r)\, J_0\left(\frac{\alpha_n r}{R}\right) dr = \hat{\varphi}(n)$$

Then we immediately can determine the inverse transform:

$$c_n = \frac{2}{R^2 J_1{}^2(\alpha_n)}\, \hat{\varphi}(n) \qquad \text{and so:}$$

$$\varphi(r) = \frac{2}{R^2} \sum_{n=1}^{\infty} \hat{\varphi}(n)\, \frac{J_0\left(\frac{\alpha_n r}{R}\right)}{J_1{}^2(\alpha_n)}\,.$$

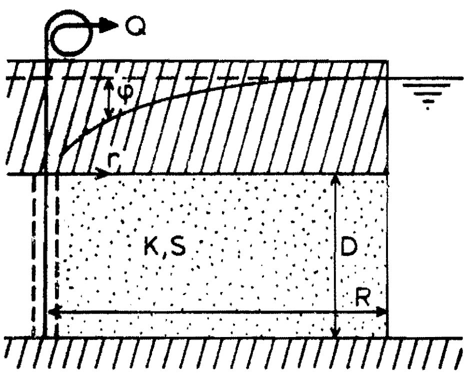
Like the infinite Hankel transformation, the finite transformation also reduces the terms $\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr}$

to the transformed function itself as follows:

$$H_n\left\{\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr}\right\} = \alpha_n J_1(\alpha_n)\,\varphi(R) - \lim_{r\to 0}\left(r\frac{d\varphi}{dr}\right) - \frac{\alpha_n^2}{R^2}\,\hat{\varphi}(n)$$

<u>Hence, application of the finite Hankel transformation is useful in axial symmetric problems with given horizontal velocity at r = o and given potential distribution at a fixed distance from the centre.</u>

For example, calculate the unsteady drawdown distribution caused by abstracting a discharge Q from a fully penetrating well at the centre of a circular island. The aquifer is assumed to be confined.



$\varphi = \varphi(r,t)$

$$\left[\begin{array}{l} \dfrac{d^2\varphi}{dr^2} + \dfrac{1}{r}\dfrac{d\varphi}{dr} = \beta^2 \dfrac{d\varphi}{dt} \qquad\qquad (\beta^2 = \dfrac{S}{kD}) \\[2ex] \varphi(r,o) = o \qquad \varphi(R,t) = o \\[2ex] \lim\limits_{r \to o} \left(r \dfrac{d\varphi}{dr}\right) = - \dfrac{Q}{2\pi kD} \quad \text{for } t > o \end{array}\right]$$

Finite Hankel transformation with respect to r gives:

$$\left[\frac{Q}{2\pi kD} - \frac{\alpha_n^2}{R^2}\hat{\varphi} = \beta^2 \frac{d\hat{\varphi}}{dt} \qquad \hat{\varphi}(n,o) = o \quad , \right]$$

an ordinary differential equation with solution:

$$\hat{\varphi}(n,t) = \frac{Q}{2\pi kD} \cdot \frac{R^2}{\alpha_n^2} \left\{ 1 - \exp\left(- \frac{\alpha_n^2 t}{\beta^2 R^2}\right) \right\} .$$

The reverse Hankel transformation yields immediately:

$$\varphi(r,t) = \frac{Q}{\pi kD} \sum_{n=1}^{\infty} \frac{J_o\left(\frac{\alpha_n r}{R}\right)}{\alpha_n{}^2 J_1{}^2(\alpha_n)} \left\{1 - \exp\left(-\frac{\alpha_n{}^2 t}{\beta^2 R^2}\right)\right\}$$

As $\sum_{n=1}^{\infty} \frac{J_o\left(\frac{\alpha_n r}{R}\right)}{\alpha_n{}^2 J_1{}^2(\alpha_n)}$ is the Fourier-Bessel representation on the interval $a \leq r \leq R$ of the function: $\frac{1}{2} \ln \frac{R}{r}$

we get:

$$\varphi(r,t) = \frac{Q}{2\pi kD} \ln \frac{R}{r} - \frac{Q}{\pi kD} \sum_{n=1}^{\infty} \frac{J_o\left(\frac{\alpha_n r}{R}\right)}{\alpha_n J_1{}^2(\alpha_n)} \exp\left(-\frac{\alpha_n{}^2 KDt}{R^2 S}\right)$$

If t tends to infinity the series vanishes and the well known solution for the steady state is obtained:

$$\varphi(r) = \frac{Q}{2\pi KD} \ln \frac{R}{r} \quad .$$

b.3. Fourier transformations

The Fourier transformations can be divided into

1e infinite sinc transformation

2e infinite cosine transformation

3e finite sine transformation

4e finite cosine transformation

These transformations are useful in solving problems which are not axial symmetric as they all eliminate the second differential quotient of a variable f,i. $\frac{d^2\varphi}{dx^2}$ or $\frac{d^2\varphi}{dz^2}$.

The operations necessary for these transformations are similar to those of the Hankel transforms and it will suffice to refer to the survey of integral transformations at the end of this paper.

In the preceding only a few examples of exact solutions for two or three-dimensional steady or non-steady problems have been given, but there are numerous problems that can be solved by means of integral transformation methods. However, most of them are not mentioned in geohydrological literature.
This is amazing, especially if one realizes that in a branch of science where problems are very similar to those in geohydrology, namely the conduction of heat in solids, multiple solutions for the most different problems are being given.

May be the reason for this must be found in a lack of the mathematical instruments which geohydrologists have at their disposal?
If such is the case, this paper may form a small contribution to a better understanding of the possibilities of solving more complicate groundwater problems analytically.

Survey of integral transformations.

	Transformation	Main property	Inverse transform.
Laplace	$L\left\{\varphi(t)\right\} = \bar{\varphi}(s)$ $= \int_0^\infty \varphi(t)e^{-st}dt$	$L\left\{\frac{d\varphi}{dt}\right\} = s\bar{\varphi} - \varphi(o)$	a. by means of operations and from tabels b. by integration in the complex plane
Fourier A. Finite 1. sine	$S_n\left\{\varphi(x)\right\} = \bar{\varphi}(n)$ $= \int_0^a \varphi(x)\sin\frac{n\pi x}{a}\,dx$	$S_n\left\{\frac{d^2\varphi}{dx^2}\right\} =$ $-(\frac{n\pi}{a})^2\breve{\varphi} + \frac{n\pi}{a}\left\{\varphi(o)-(-1)^n\varphi(a)\right\}$	$S_n^{-1}\left\{\breve{\varphi}(n)\right\} = \varphi(x)$ $= \frac{2}{a}\sum_{n=1}^{\infty}\breve{\varphi}(n)\sin\frac{n\pi x}{a}$
2. cosine	$C_n\left\{\varphi(x)\right\} = \breve{\varphi}(n)$ $= \int_0^a \varphi(x)\cos\frac{n\pi x}{a}\,dx$	$C_n\left\{\frac{d^2\varphi}{dx^2}\right\} =$ $-(\frac{n\pi}{a})^2\breve{\varphi} + (-1)^n\frac{d\varphi}{dx}(a) - \frac{d\varphi}{dx}(o)$	$C_n^{-1}\left\{\breve{\varphi}(n)\right\} = \varphi(x) =$ $\frac{1}{a}\breve{\varphi}(o) + \frac{2}{a}\sum_{n=1}^{\infty}\breve{\varphi}\cos\frac{n\pi x}{a}$
B. Infinite 1. sine	$S\left\{\varphi(x)\right\} = \hat{\varphi}(\alpha)$ $\int_0^\infty \varphi(x)\sin(\alpha x)dx$	$S\left\{\frac{d^2\varphi}{dx^2}\right\} =$ $-\alpha^2\hat{\varphi} + \alpha\varphi(o)$ with $\varphi(\infty) = \frac{d\varphi}{dx}(\infty) = o$	$S^{-1}\left\{\hat{\varphi}(\alpha)\right\} = \varphi(x)$ $= \frac{2}{\pi}\int_0^\infty \hat{\varphi}(\alpha)\sin(x\alpha)d\alpha$
2. cosine	$C\left\{\varphi(x)\right\} = \hat{\varphi}(\alpha) =$ $\int_0^\infty \varphi(x)\cos(\alpha x)dx$	$C\left\{\frac{d^2\varphi}{dx^2}\right\} =$ $-\alpha^2\hat{\varphi} - \frac{d\varphi}{dx}(o)$ with $\varphi(\infty) = \frac{d\varphi}{dx}(\infty) = o$	$C^{-1}\left\{\hat{\varphi}(\alpha)\right\} = \varphi(\alpha)$ $= \frac{2}{\pi}\int_0^\infty \varphi(\alpha)\cos(x\alpha)d\alpha$

	Transformation	Main property	Inverse transform.
Hankel A. Finite	$H_n\left\{\varphi(r)\right\} = \hat{\varphi}(n)$ $\int_0^R r\varphi(r)\ J_o\ (\frac{\alpha_n r}{R})\ dr$ with α_n the roots of $J_o\ (\alpha) = o$	$H_n\left\{\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr}\right\} =$ $\alpha_n\ J_1(\alpha_n)\varphi(R) - \lim_{r\to o} r\frac{d\varphi}{dr}$ $- \frac{\alpha_n^2}{R^2}\hat{\varphi}(n)$	$H_n^{-1}\left\{\hat{\varphi}(n)\right\} = \varphi(r) =$ $\frac{2}{R^2}\sum_{n=1}^{\infty}\hat{\varphi}(n)\ \frac{J_o(\frac{\alpha_n r}{R})}{J_1^2(\alpha_n)}$
B. Infinite	$H\left\{\varphi(r)\right\} = \hat{\varphi}(\alpha)$ $= \int_o^{\infty} r\varphi(r)\ J_o(\alpha r)dr$	$H\left\{\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr}\right\} =$ $- \lim_{r\to o} r\frac{d\varphi}{dr} - \alpha^2\varphi$ with $\varphi\ (\infty) = \frac{d\varphi}{dr}\ (\infty) = o$	$H^{-1}\left\{\hat{\varphi}(\alpha)\right\} = \varphi(r)$ $= \int_o^{\infty}\alpha\ \hat{\varphi}\ J_o(r\alpha)d\alpha$